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Effect of non-uniform currents and depth variations upon steady discharges in shallow water

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When advection dominates diffusion, there are special directions (rays), distinct from but closely related to the contaminant flux vector, along which information is carried. The geometry of these ray paths is found to depend in a simple way upon the advection–diffusion vector $\frac{1}{2}\mathbf{u}/D$, where $\mathbf{u}$ is the flow velocity and D the cross-stream diffusivity. Simple models are used to show that for a steady point discharge the contaminant concentration is greatest in shallow water and towards the outside of bends.

1. Introduction

Kay (1982) has demonstrated the strong influence of depth topography upon the horizontal spreading of a contaminant plume in vertically well-mixed currents. His mathematical analysis used the method of images, with real and virtual contaminant sources, and is therefore restricted to certain abrupt geometries. The purpose of the present work is to investigate the complementary problem of a steady discharge in water of smoothly varying depth.

For turbulent currents in shallow water, the effective diffusivity for the spreading of contaminants scales as the product of the local water depth and the friction velocity (Elder 1959). Typically the friction velocity u_* is between ten and twenty times smaller than the bulk velocity $\mathbf{u}$. Thus, there is an implicit small parameter

$$\epsilon = u_*/|\mathbf{u}| \tag{1.1}$$

in the equation for horizontal dispersion.

The idea of 'ray methods' is to exploit explicitly the presence of such a small parameter. Originally these methods were developed for wave problems. However, Cohen & Lewis (1967) have shown that in principle these same mathematical methods can be used for diffusion problems. Here we apply these methods to the two-dimensional advection–diffusion equation. The outcome is a simple constructive procedure, which yields an accurate approximation to the concentration distribution.

2. The ray ansatz

As the starting point for our mathematical analysis, we take the horizontal dispersion equation to be of the form

$$\nabla.(hc\mathbf{u}) - \epsilon\nabla.(hD\nabla c) = 0. \tag{2.1}$$

Here ∇ is the horizontal gradient operator (∂_x, ∂_y), $h(x,y)$ is the water depth, $\mathbf{u}(x,y)$

the vertically averaged flow velocity, $c(x, y)$ the concentration, and $D(x, y)$ the turbulent diffusivity across the flow. Strictly, we should use a diffusivity tensor D_{ij} because there is a considerable disparity between the dispersion coefficients along and across the flow (Elder 1959). However, away from the immediate vicinity of the discharge this technical complication does not significantly affect the contaminant distribution (i.e. concentration gradients are predominately across the flow). The parameter ϵ is a reminder of the relative importance of the terms in (2.1). For example, if we introduced advection-velocity and length scales U and L, then the non-dimensional diffusivity D/UL would be numerically small.

For a uniform medium we know that the exact solution for a steady point discharge rapidly attains an exponential profile across the contaminant plume (Kay 1982). Moreover, at distances of order unity from the discharge the exponent has a magnitude ϵ^{-1}. When the depth topography and current are non-uniform, we assume that these properties still hold. Thus we pose the representation

$$c = a \exp(\pm \phi/\epsilon) \tag{2.2}$$

(Cohen & Lewis 1967, equation (3.1.1)). Here $a(x, y; \epsilon)$ is an amplitude factor and $\phi(x, y; \epsilon)$ the decay exponent. A minor departure from the method of Cohen & Lewis (1967) is in the use of the $\pm$ sign. If the steady flow were reversed (i.e., $+u$ replaced by $-u$), then the direction of the plume would likewise be reversed. The $\pm$ sign is a technical device to ensure that this property is preserved exactly.

3. Eikonal and transport equations

If we substitute the ansatz (2.2) into the advection–diffusion equation (2.1), with flow velocity $\pm \mathbf{u}$, then we generate terms proportional to

$$\exp(\pm \phi/\epsilon) \quad \text{and} \quad \pm \exp(\pm \phi/\epsilon).$$

Equating these groups of terms separately to zero, we have

$$h\mathbf{u} \cdot \nabla\phi - hD(\nabla\phi)^2 = \epsilon^2 \nabla \cdot (hD\nabla a)/a, \tag{3.1}$$

and

$$\nabla \cdot (h\mathbf{u}a) - \nabla \cdot (hDa\nabla\phi) - hD\nabla\phi \cdot \nabla a = 0. \tag{3.2}$$

Cohen & Lewis (1967, equations (3.1.2) and (3.1.3)) use a different splitting into two equations, with the magnitude of the right-hand-side terms increased from ϵ^2 to ϵ.

As yet there have been no approximations made. By the nonlinear transformation (2.2), we have replaced the original linear equation (2.1) by the coupled nonlinear equations (3.1) and (3.2). The crucial simplification is that, unlike the original variable $c(x, y)$, the new variables ϕ and a vary slowly. Thus, when ϵ is small, we can infer that the variables have asymptotic expansions

$$\phi = \phi_0 + \epsilon^2\phi_1 + \ldots, \quad \text{and} \quad a = a_0 + \epsilon^2 a_1 + \ldots. \tag{3.3}$$

The rapid reduction in size of the successive terms in these expansions means that in practice it is not necessary to proceed beyond the leading terms. Thus, we shall omit the zero subscript and simply replace (3.1) by the first-order equation

$$(\mathbf{u} - D\nabla\phi) \cdot \nabla\phi = 0. \tag{3.4}$$

We note that, as a result of the expansion in ϵ, the equations for the decay exponent

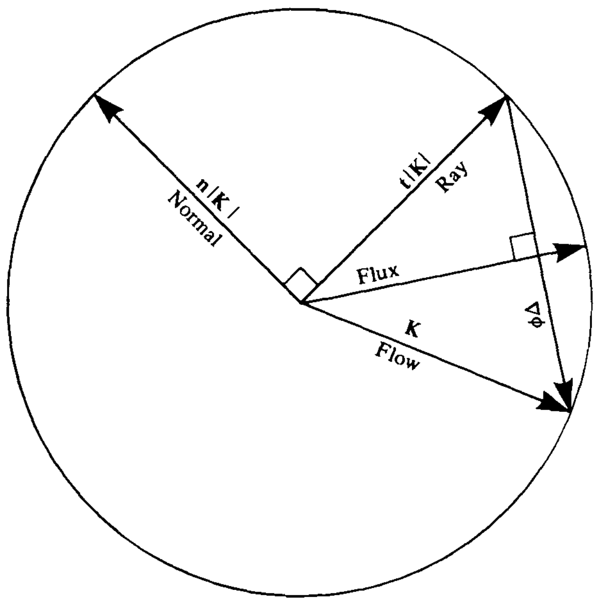

FIGURE 1. Geometrical relationships between the directions of the current, the contaminant flux and the ray paths.

and the amplitude factor have become decoupled. In the terminology of ray methods, (3.4) is called the 'eikonal' equation and (3.2) the 'transport' equation.

To a first approximation, the concentration gradient ∇c and the contaminant flux vector $\mathbf{u}c - \epsilon D\nabla c$ are given by

$$[\epsilon^{-1}a \exp(\phi/\epsilon)]\nabla\phi \quad \text{and} \quad [a \exp(\phi/\epsilon)](\mathbf{u} - D\nabla\phi). \tag{3.5}$$

Thus, the eikonal equation admits of the physical interpretation that the contaminant flux is orthogonal to the direction of maximum concentration gradient. Equivalently, the concentration distribution has arisen because of the flux of contaminant and therefore the concentration gradient is minimised in that direction. As is familiar in wave problems, 'information' or changes in the solution are carried along yet another vector field – the ray paths.

4. Ray paths

In order to solve (3.4), we shall use the method of characteristics or rays (Courant & Hilbert 1962, chapter 2). The ray direction is along the unit vector

$$\mathbf{t} = (\mathbf{u} - 2D\nabla\phi)/|\mathbf{u}| \tag{4.1}$$

(see figure 1). Let s be the arc length along a ray path. The derivative of a function $f(x,y)$ with respect to s is defined by

$$\partial f/\partial s \equiv (\mathbf{t}.\nabla)f = (\mathbf{u} - 2D\nabla\phi).\nabla f/|\mathbf{u}|. \tag{4.2}$$

In particular, taking $f = \phi$ and eliminating $(\nabla\phi)^2$ by means of the Eikonal equation (3.4), we obtain the result

$$\partial\phi/\partial s = -(|\mathbf{u}| - \mathbf{u}.\mathbf{t})/2D. \tag{4.3}$$

Thus, the rate of exponential decay along a ray depends upon the relative direction of the ray path and the current.

In order to extend a ray path away from the source we must know both its direction $\mathbf{t}$ and its curvature κ. If $\mathbf{k}$ is the unit vertical vector, then we define the ray normal $\mathbf{n}$ so that $\mathbf{t}$, $\mathbf{n}$, $\mathbf{k}$ form a right-handed triad (see figure 1). In terms of $\mathbf{n}$, s and t, the curvature κ is defined as

$$\kappa \equiv \mathbf{n}.\frac{\partial}{\partial s}\mathbf{t} = \mathbf{n}.[(\mathbf{t}.\nabla)t] = \mathbf{n}.[(\nabla\times\mathbf{t})\times\mathbf{t}], \tag{4.4}$$

since $\mathbf{t}.\mathbf{t} = 1$. If we define the advection–diffusion vector $\mathbf{K}$ by

$$\mathbf{K} = \mathbf{u}/2D \tag{4.5}$$

(Kay 1982), then it follows from (4.1) that

$$\mathbf{t} = (\mathbf{K}-\nabla\phi)/|\mathbf{K}|. \tag{4.6}$$

Hence, with some manipulation of vector products, it follows from (4.4) and (4.6) that

$$\kappa = (\mathbf{n}.\nabla|\mathbf{K}|+\mathbf{k}.(\nabla\times\mathbf{K}))/|\mathbf{K}|. \tag{4.7}$$

Thus, rays tend to go into regions in which $|\mathbf{K}|$ is large and are bent in the sense of rotation of $\mathbf{K}$ (see §6).

5. Amplitude factor

To complete our solution for the concentration distribution $c(x,y;\epsilon)$, we need to determine the amplitude factor $a(x,y)$. Again, this is facilitated by the use of ray paths. First, we observe that by using (4.2) the transport equation (3.2) can be written as an ordinary differential equation:

$$\partial a/\partial s + a[\nabla.(h\mathbf{u})-\nabla.(hD\nabla\phi)]/h|\mathbf{u}| = 0. \tag{5.1}$$

Next, we introduce a parameter p which labels the individual rays, and we consider the ray separation

$$J(p,s) = \left|\frac{\partial(x,y)}{\partial(p,s)}\right|. \tag{5.2}$$

If we take the s-derivative of this formula and eliminate second derivatives with respect to p by means of the chain rule, for example,

$$\frac{\partial^2 x}{\partial s\,\partial p} = \left(\frac{\partial x}{\partial p}\frac{\partial}{\partial x}+\frac{\partial y}{\partial p}\frac{\partial}{\partial y}\right)\frac{\partial x}{\partial s}, \tag{5.3}$$

then we can obtain the general result (Cohen & Lewis 1967, equation (3.1.13))

$$\partial J/\partial s = J\nabla.\mathbf{t}. \tag{5.4}$$

For the particular case being studied here, with the ray direction $\mathbf{t}$ given by (4.1), this result can be written

$$\partial J/\partial s = J[\nabla.(\mathbf{u}/|\mathbf{u}|)-2\nabla.(D\nabla\phi/|\mathbf{u}|)]. \tag{5.5}$$

Using (4.1) in the form $D\nabla\phi = \frac{1}{2}(\mathbf{u}-|\mathbf{u}|\mathbf{t})$, we can now re-write the transport equation (5.1) as

$$\frac{1}{a}\frac{\partial a}{\partial s}+\frac{1}{2}\frac{1}{J}\frac{\partial J}{\partial s}+\frac{1}{2}\frac{1}{h|\mathbf{u}|}\frac{\partial}{\partial s}(h|\mathbf{u}|) = -\frac{1}{2}\frac{\nabla.(h\mathbf{u})}{h|\mathbf{u}|}. \tag{5.6}$$

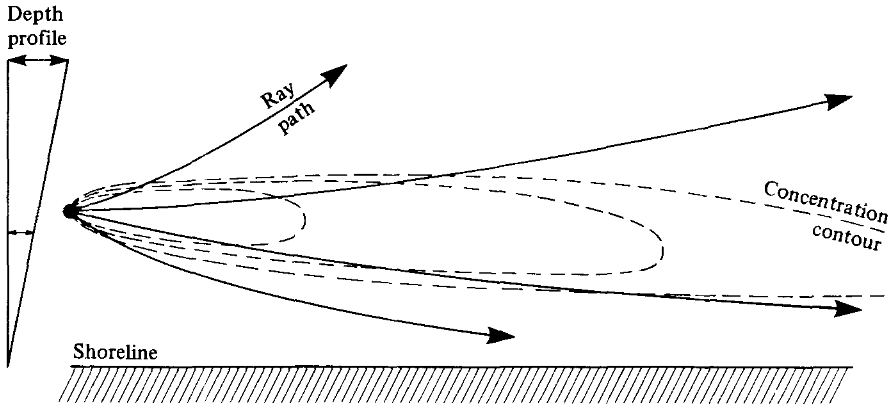

FIGURE 2. Ray paths for contaminant spreading when the water depth varies across the flow.

Conservation of mass for the steady current implies that the right-hand side is zero. Thus, (5.6) can be integrated at sight to yield the important result

$$aJ^{\frac{1}{2}}h^{\frac{1}{2}}|\mathbf{u}|^{\frac{1}{2}} = \text{constant along rays.} \tag{5.7}$$

An immediate implication is that the contaminant concentrations are greatest where the rays are convergent (i.e. where the ray separation J is relatively small).

6. Illustrative examples

For any real topography, the ray curvature would have to be calculated numerically. However, it is of interest to consider some simple examples.

For steady, unstratified, plane parallel flow in water of non-uniform depth $h(y)$, the velocity varies as $h^{\frac{1}{2}}$ and the turbulent diffusivity as $h^{\frac{3}{2}}$ (Smith 1976). To model this we take

$$\mathbf{K} = \text{constant}\,(1/h, 0, 0). \tag{6.1}$$

From (4.4) it then follows that

$$\kappa = (1-t_1)\,h'/h, \tag{6.2}$$

where t_1 is the x-component of the unit vector $\mathbf{t}$. Thus, in accord with Kay's (1982) analysis, the rays, and hence the contaminant flux, tend to curve towards the deeper water (see figure 2). Moreover, the variation in $h|\mathbf{u}|$ further exaggerates the asymmetry of the concentration distribution. For more complicated velocity and diffusivity distributions, the same general features can be expected provided that $\mathbf{K}$ increases towards the shoreline.

As a complementary example, we next consider the case in which the depth variations take place in the direction of flow. The velocity varies as h^{-1} and the diffusivity remains constant:

$$\mathbf{K} = \text{constant}\,(1/h(x), 0, 0). \tag{6.3}$$

If we use the component notation $\mathbf{t} = (t_1, t_2, 0)$, then it follows that

$$\kappa = t_2 h'/h, \tag{6.4}$$

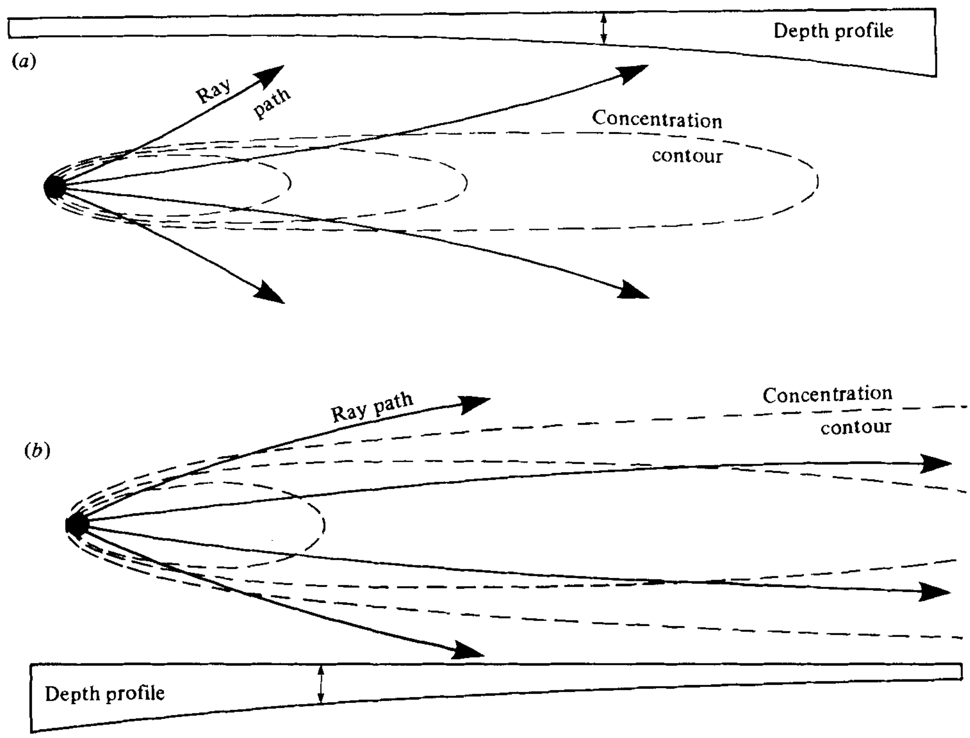

FIGURE 3. Ray paths for contaminant spreading when the water depth varies along the flow.

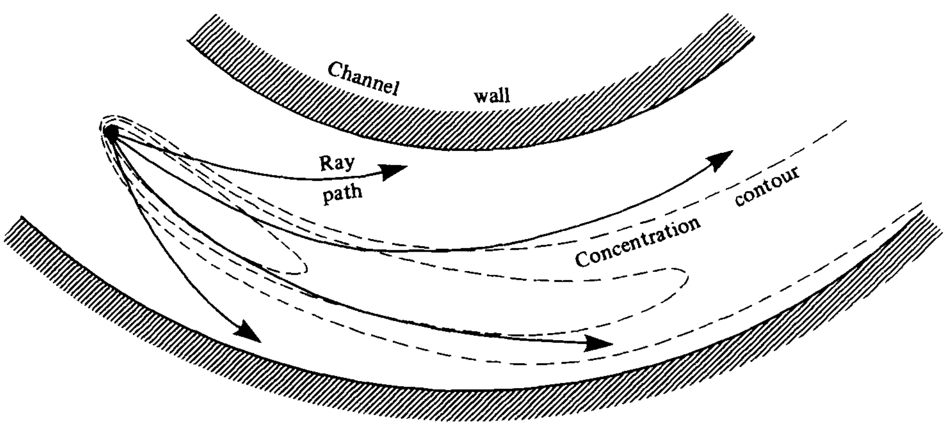

FIGURE 4. Ray paths for a source in a curved channel.

and if the flow is into deeper water, then the rays are divergent (see figure 3). This implies that the contaminant is more spread-out with correspondingly reduced concentration.

In cylindrical polar co-ordinates, the advection–diffusion vector for a curved channel takes the form

$$\mathbf{K} = \text{constant}\,(0, -1/h(r), 0). \tag{6.5}$$

Using the standard formulae for ∇ and $\nabla\times$ in this axis system, we obtain

$$\kappa = -1/r + (1+t_\theta)\, h'/h. \tag{6.6}$$

Hence, in the absence of depth variations, the curvature of the rays is the same as the curvature of the current. This has the nice consequence that the central ray from the contaminant source continues along the changing flow direction. However, to the outside of the bend the rays are more closely aligned with the current (see figure 4), implying that the contaminant distribution will be skewed towards this side (i.e. the concentration exhibits the inertia-like tendency to move in a straight line).

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Appendix. Accuracy of the solutions

One class of problems for which it is possible to solve the advection–diffusion equation (1.1) exactly is obtained when the water depth and diffusivity are taken to be constants and the flow velocity is irrotational:

$$u_1 = \frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad u_2 = \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}. \tag{A 1}$$

Here $X(x,y)$ is a velocity potential and $Y(x,y)$ a stream function. A change of variables to X, Y transforms (1.1) to the Cartesian form

$$\frac{\partial c}{\partial X} - \epsilon D\left[\frac{\partial^2 c}{\partial X^2} + \frac{\partial^2 c}{\partial Y^2}\right] = 0. \tag{A 2}$$

For a steady point discharge, the exact solution is

$$c = \exp(X/2\epsilon D)\, K_0([X^2+Y^2]^{\frac{1}{2}}/2\epsilon D). \tag{A 3}$$

For large values of the argument, the modified Bessel function has the asymptotic approximation

$$K_0(z) \sim (\pi/2z)^{\frac{1}{2}} \exp(-z). \tag{A 4}$$

At $z = 1$ the error is less than 10%, and by $z = 10$, the error is down to 1%. The presence of the small parameter ϵ in the exact solution (A 3) means that the asymptotic form is achieved very close to the source. As we might have anticipated, the asymptotic solution for c is precisely the ray solution

$$c \sim \frac{(\pi/\epsilon D)^{\frac{1}{2}}}{[X^2+Y^2]^{\frac{1}{4}}} \exp\left(-\frac{1}{2\epsilon D}\{[X^2+Y^2]^{\frac{1}{2}} - X\}\right). \tag{A 5}$$

Back in the physical (x,y) co-ordinates, the condition for one-per-cent accuracy is that the distance from the source exceeds

$$20\,\epsilon D/|\mathbf{u}|. \tag{A 6}$$

For turbulent open-channel flow this requirement is actually less stringent than the conditions for (1.1) to be applicable (i.e. for the dispersion to be primarily horizontal and not vertical).

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